IV 3-manifolds

A. Heegaard splittings

let $H_g = (0\text{-handle}) \cup g\,(1\text{-handles})$ s.t. H_g is oriented

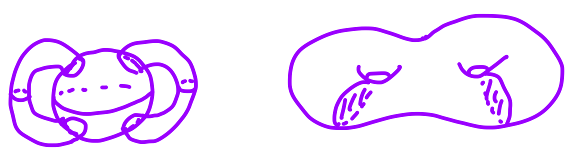
this is called a genus g handlebody

notice that since any embedding of $S^0 \hookrightarrow S^2$ is isotopic to any other

attaching sphere of 1-handle

and $\exists$ unique way to "frame" 1-handle to get oriented mfd

we see H_g only depends on g



exercise: 1) Show an oriented 3-mfd M is a handlebody of genus g

$\Leftrightarrow$

$\exists\, g$ disjoint embedded disks $D_1, \dots, D_n$

s.t. $M \setminus \bigcup D_i \cong B^3$

2) If Σ is an oriented surface w/ $\partial \neq \emptyset$, then $\Sigma \times [0,1]$ is a handlebody.

Given a closed oriented 3-manifold M^3, it has a handle decomposition with one 0-handle and one 3-handle

can assume all the 1-handles come before 2-handles

let $\Sigma \subset M$ be ∂W_1

<u>note:</u> W_1 is a handle body of genus $g = \#$ 1-handles and Σ genus g sfc

similarly $W' = \overline{M - W_1}$ turned upside down is a handle body of genus $g' = \#$ 2-handles

since $\partial W_1 = \Sigma = \partial W'$ we see $g = g'$

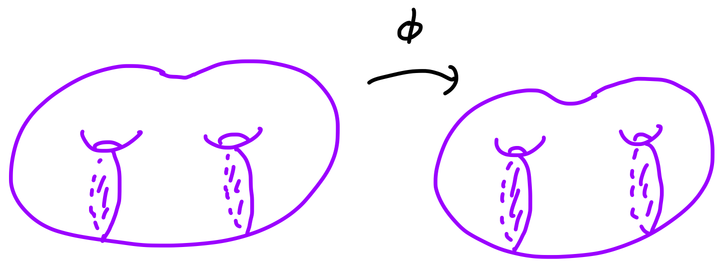
and M is the union of 2 genus g handle bodies!

Σ called a <u>Heegaard</u> surface for M and $M \setminus \Sigma$ a <u>Heegaard splitting</u>

$\exists$ embeddings $H_g \xrightarrow{\psi_1} M$, $H_g \xrightarrow{\psi_2} M$ for cpts of $M \setminus \Sigma$

let $\phi = \psi_2|_{\partial H_g}^{-1} \circ \psi_1^{-1}|_{\partial H_g} : \Sigma_g \to \Sigma_g$

this is a diffeo. and



$$M \cong H_g^1 \cup H_g^2 / p \in \partial H_g^1 \sim \phi(p) \in \partial H_g^2$$

so any 3-mfd obtained by gluing 2 handlebodies together!

really simple manifolds!

so <u>all 3-mfds described by diffeos of surfaces</u>!

So we have 2 notions of a Heegaard splitting of M^3

1) a Heegaard splitting of M is a surface $\Sigma \subset M$ such that $M \setminus \Sigma = 2$ handlebodies V_0 and V_1

Σ is called the Heegaard surface

2) an (abstract) Heegaard splitting of M is an orientation reversing diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ of a genus g surface and a diffeomorphism

$$f : H_g \cup_\phi H_g \longrightarrow M$$

note: from above 1) gives 2)

and given 2), $f(\partial H_g) = \Sigma$ is a splitting as in 1)

but given ϕ in 2) there can be different f in 2)

so 2) does not (necessarily) define a unique Σ as in 1)

another way to describe 3D handle decompositions "Kirby pictures" A.K.A. "Heegaard Diagrams"

just as for surfaces two 3-manifolds M_1, M_2 are diffeomorphic $\Leftrightarrow$ $\widehat{M}_1, \widehat{M}_2$ are diffeomorphic where $\widehat{M}_i = \overline{M_i - B^3}$

this is not true in all dimension but is true here (non-trivial fact due to Smale)

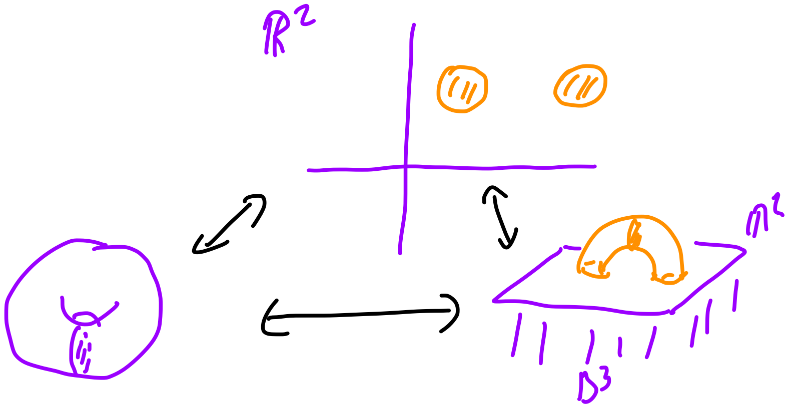
so can understand a 3-mfd M by looking at

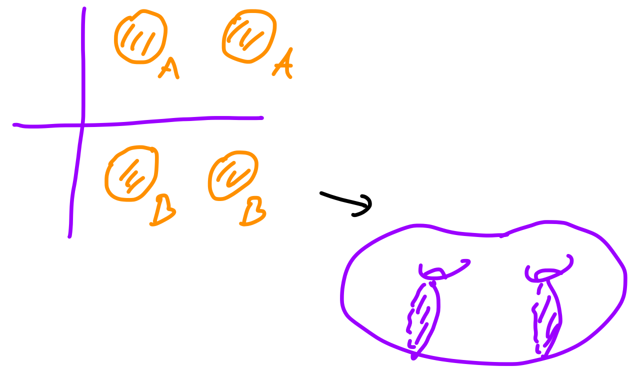
$$M^2 = (0\text{-handle}) \cup k(1\text{-handles}) \cup k(2\text{-handles})$$

handles attached to $\partial(0\text{-handle}) = S^2$

and attaching regions of handles can be assumed to be disjoint from a pt in S^2, ie in $\mathbb{R}^2 = S^2 - \{pt\}$

example: attaching region of 1-handle is $S^0 \times D^2$

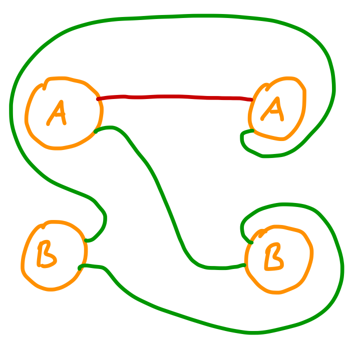




attaching sphere of 2-handle is S^1 (and attaching region uniquely determined by this $\pi_1(O(1)) = \{e\}$, just thicken S^1)

example:

1)

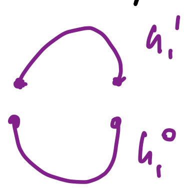


exercise: this is $\mathbb{R}P^3 = S^3/$identify antipode

Hint: See below

let's work through some examples

$S^1 \times S^2$: note: $S^1 =$

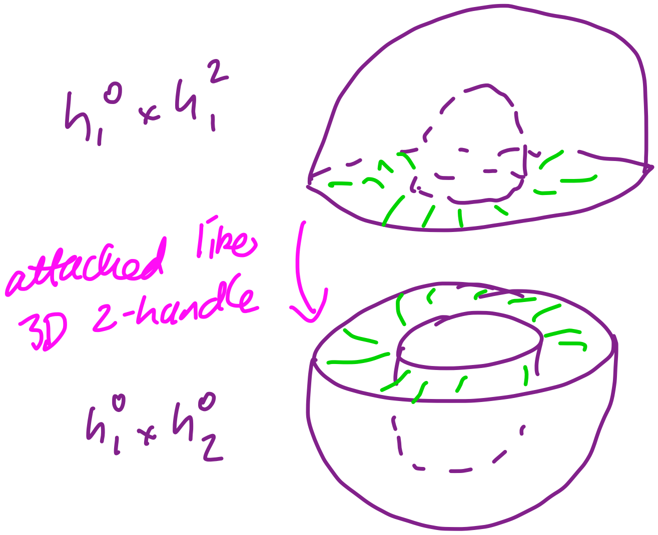


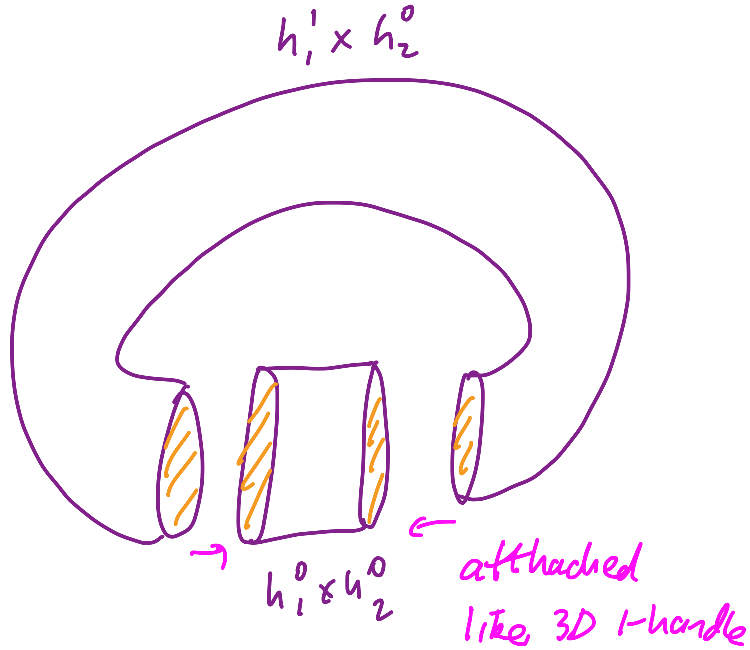
$S^2 =$

$$S^1 \times S^2 = (h_1^0 \cup h_1^1) \times (h_2^0 \cup h_1^2)$$
$$= (h_1^0 \times h_2^0) \times (h_1^1 \times h_2^0) \times (h_1^1 \times h_2^0) \times (h_1^1 \times h_1^2)$$

we will see: 3D 0-handle, 3D 1-handle, 3D 2-handle, 3D 3-handle

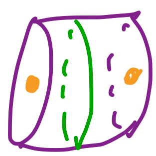
so $S^1 \times h_2^0$ is



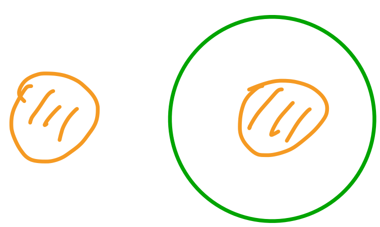


finally $h_1^1 \times h_1^2 = D^3$ glued along ∂D^3 so 3D 3-handle

if we ignor 3-handle then 0-handle is as follows with attaching spheres of 1- and 2-handle indicated



so the Heegaard diagram is



lens spaces: recall a lens space is the result of gluing two $S^1 \times D^2$ together by a diffeomorphism of their boundary

$$\phi : \underbrace{\partial(S^1 \times D^2)}_{= T^2} \to \underbrace{\partial(S^1 \times D^2)}_{= T^2}$$

Fact: $\pi_0(\mathrm{Diff}(T^2)) = GL(2;\mathbb{Z})$

note: $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ $\left((x,y)\sim(x',y') \text{ if } \begin{smallmatrix} x-x' \\ y-y' \end{smallmatrix} \in \mathbb{Z}\right)$

if $A \in GL(2;\mathbb{Z})$, then $A(\mathbb{Z}^2) = \mathbb{Z}^2$

so A induces a map on T^2

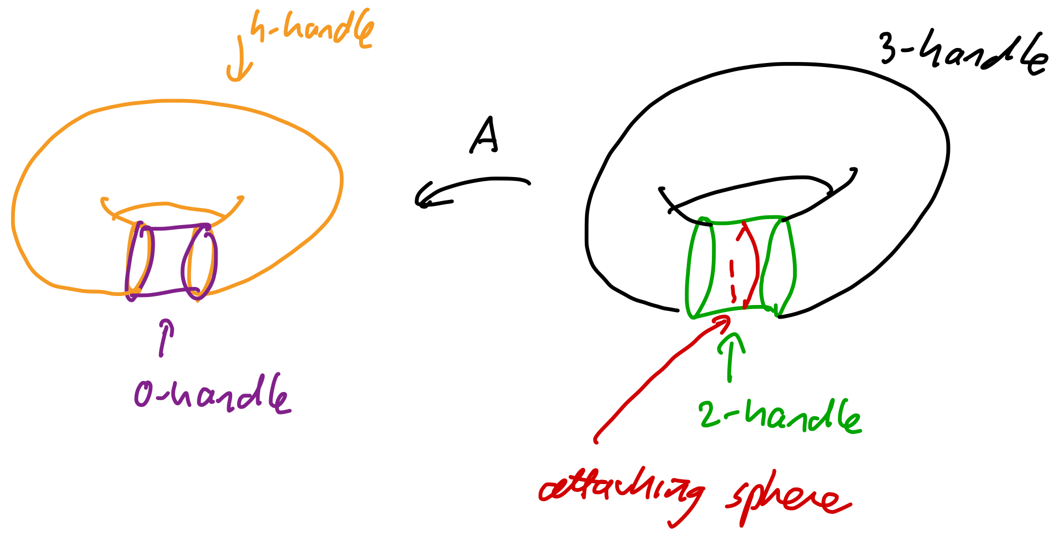
since A is inventable the above map is too, $\therefore$ diffeo

now given $p,q \in \mathbb{Z}$ relatively prime

$\exists\, r,s$ s.t. $qr-ps = -1$ and $A = \begin{bmatrix} r & p \\ s & q \end{bmatrix} \in SL(2;\mathbb{Z})$

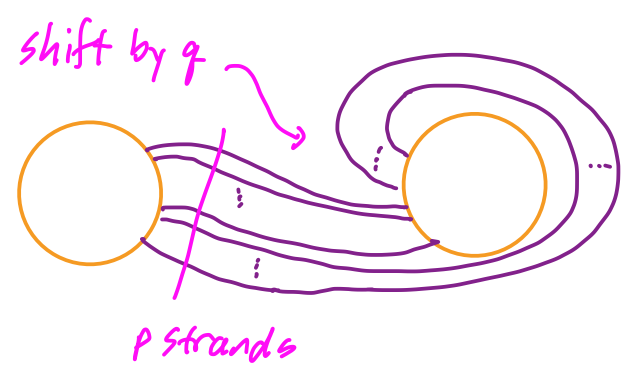
note: $S^1 \times D^2 = (0\text{-}h) \cup (1\text{-}h)$

if $S^1\times D^2$ is second solid torus, then since we are gluing it to the above torus we think of it as a 2-handle and 3-handle



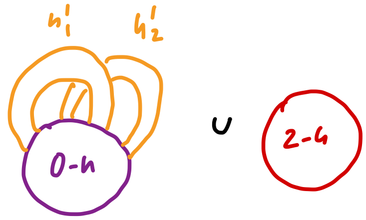
A sends the attaching sphere above to a (p,q)-curve

so the Heegaard diagram is



$T^2 \times I$:

recall $T^2 =$

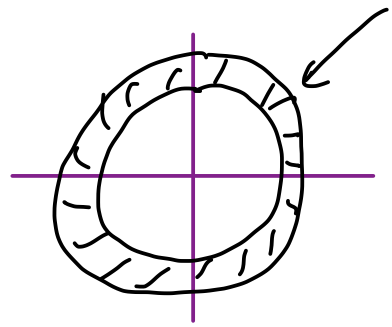


thicken this by I we see

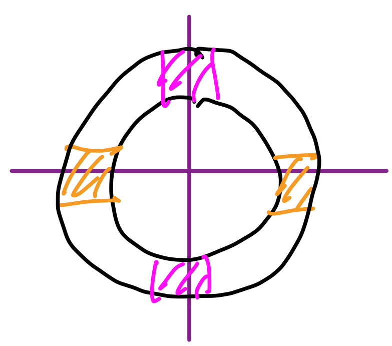
the Heegaard diagram is

∂(0-handle)

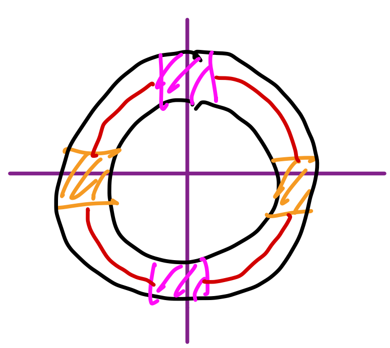
this is $(\partial(0\text{-}h)) \times I$



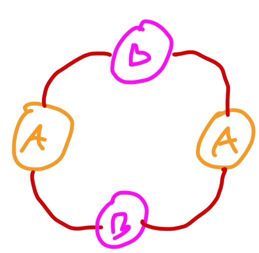
so thickend 1-handles are 3D 1-handles



and thickend 2-handle is 3D 2-handle



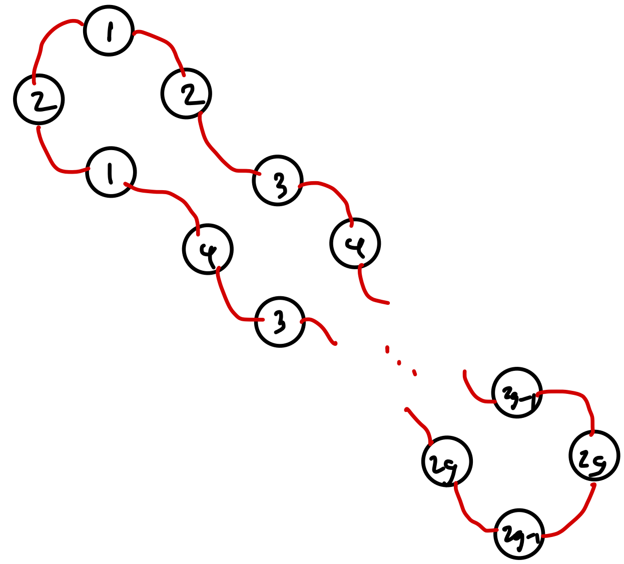
so diagram is



exercise: $\partial(T^2\times[0,1]) = T^2\times\{0,1\}$

see these 2 tori in the above diagram

exercise: show $\Sigma_g\times I$ has the following Heegaard diagram

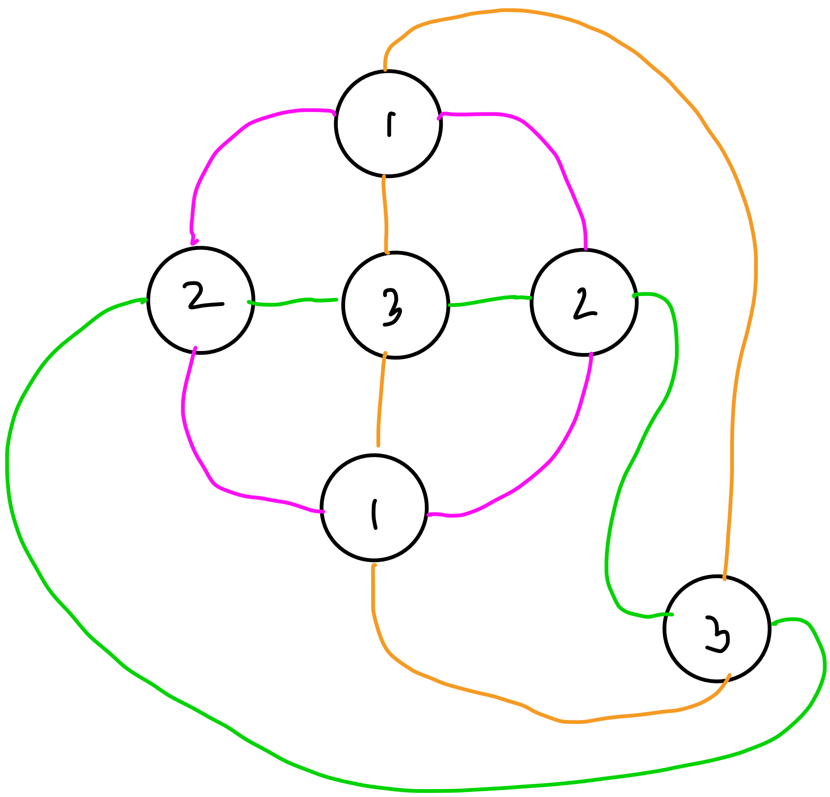


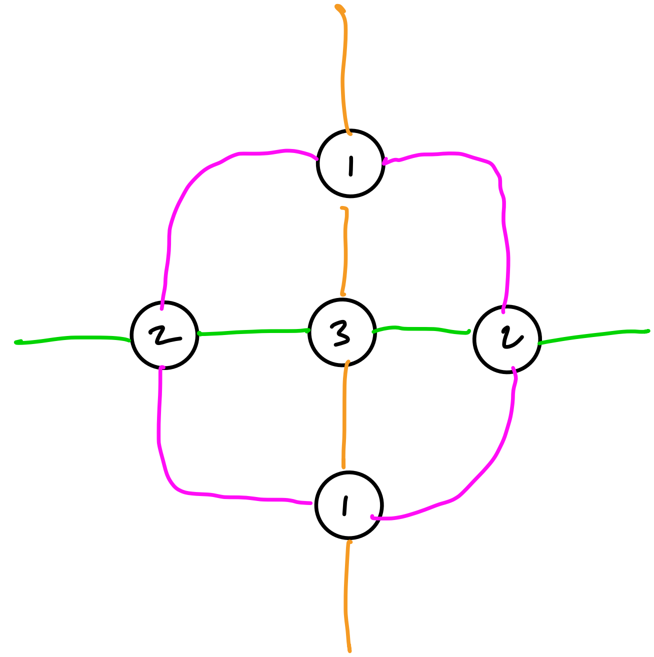
T^3:

$$T^3 = T^2\times S^1 = (h^0\cup h_1^1\cup h_2^1\cup h^2)\times(\bar h^0\cup \bar h^1)$$

$$= \underbrace{(T^2\times \bar h^0)}_{\text{example above}}\cup(\underbrace{h^0\times\bar h^1}_{1\text{-}h}\cup \underbrace{h_1^1\times\bar h^1}_{2\text{-}h}\cup \underbrace{h_2^1\times\bar h^1}_{2\text{-}h}\cup \underbrace{h^2\times\bar h^1}_{3\text{-}h})$$

similarly for $h_2^1\times\bar h^1$





exercise: See $S^1 \times S^1 \times \{pt\}$, $S^1 \times \{pt\} \times S^1$, $\{pt\} \times S^1 \times S^1$ in picture above

exercise: Draw a Heegaard diagram of $\Sigma_g \times S^1$

Surface bundles over S^1

if M fibers over the circle $\begin{array}{c} M \\ \downarrow \pi \\ S^1 \end{array}$

then $\pi^{-1}(\theta) = \Sigma$ and $M \setminus \Sigma$ is a Σ-bundle over $[0,1]$ so

$$\begin{array}{c} M \setminus \Sigma = \Sigma \times [0,1] \\ \downarrow \\ [0,1] \end{array}$$

and to recover M we glue $\Sigma \times \{1\}$ to $\Sigma \times \{0\}$ by a diffeomorphism

$$\phi : \Sigma \to \Sigma$$

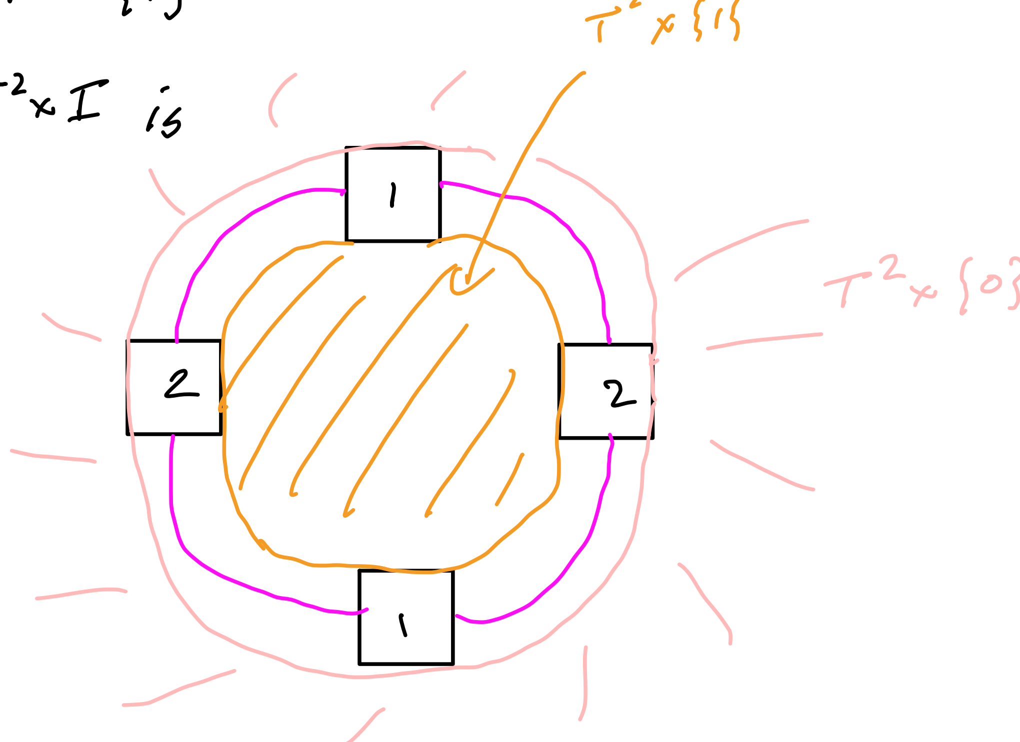
called the monodromy

that is $M \cong \Sigma \times [0,1] / (x,1) \sim (\phi(x), 0)$

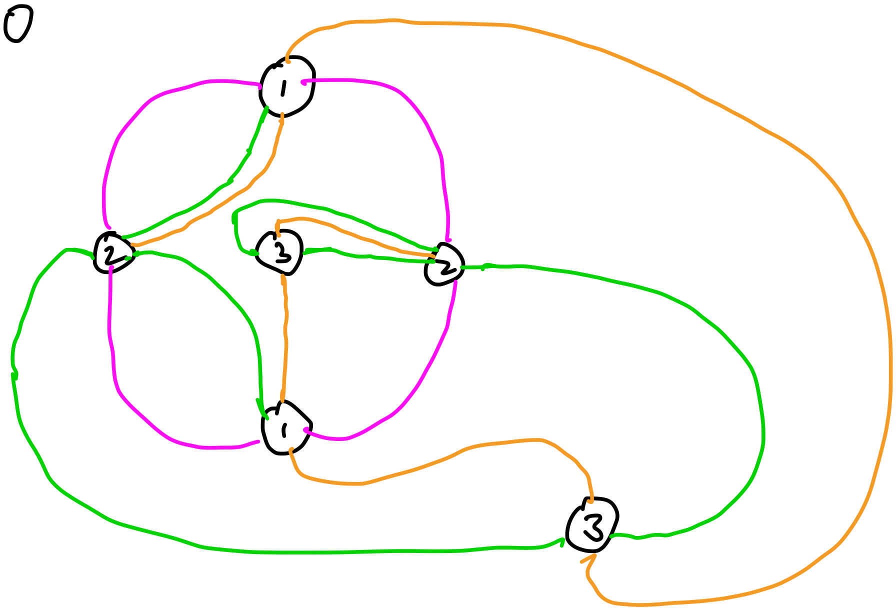
so what is a Heegaard diagram for M?

if $\Sigma = T^2$, the we proceed as above for T^3 (which is a T^2-bundle with $\phi = id_{T^2}$) but when we attach the last 2-handles we apply ϕ to the attaching curve on $T^2 \times \{1\}$

recall $T^2 \times I$ is



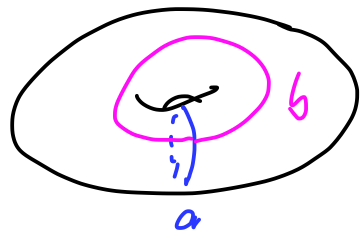
so



this is the T^2-bundle with monodromy

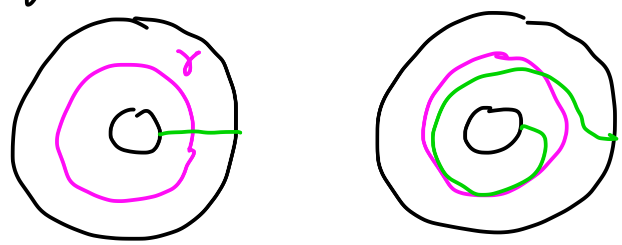
$$\phi = \tau_a \circ \tau_b$$

where



and τ_γ is a Dehn twist about γ

on a nbhd of γ
τ_γ is



$$(\theta, t) \longmapsto (\theta - 2\pi t, t)$$

and τ_γ is the identity outside the neighborhood

exercise: 1) compute the homology and π_1 of the manifold above

2) identify the manifold above in a different way

Hint: maybe wait till later when we have other ways to describe 3-mfds

exercise: draw some nontrivial surface bundles and compute their algebraic topology

(for a surface of genus g what is the largest and smallest b_1 you can realize?)

S^1-bundles over a surface:

given an S^1-bundle over a surface of genus g

$$\begin{array}{ccc} S^1 & \longrightarrow & Y \\ & & \downarrow \\ & & \Sigma \end{array}$$

then let $D^2 \subset \Sigma$ be a disk

clearly $Y|_{D^2}$ is an S^1-bundle over D^2

so $Y|_{D^2} \cong D^2 \times S^1$ bundles over contractible spaces are trivial

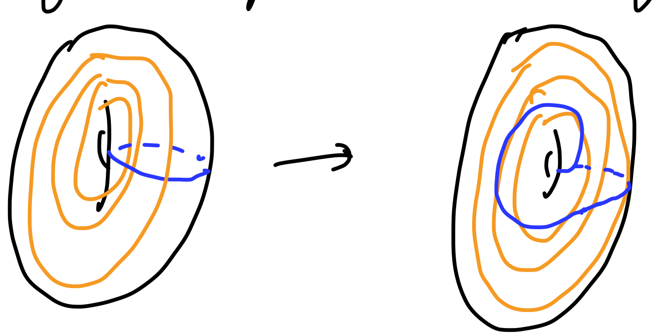
exercise: if Y is orientable then $Y|_{\Sigma \setminus D^2} \cong (\Sigma \setminus D^2) \times S^1$

so $Y \cong (D^2 \times S^1) \cup ((\Sigma \setminus D^2) \times S^1)$ with

$\partial D^2 \times S^1$ identified to $\partial(\Sigma \setminus D^2) \times S^1$ by a diffeomorphism taking S^1-fibers to S^1-fibers

diffeos of S^1 are isotopic to $O(2)$

so we need an element of $\pi_1(O(2)) \cong \mathbb{Z}$ to glue the pieces above to get Y

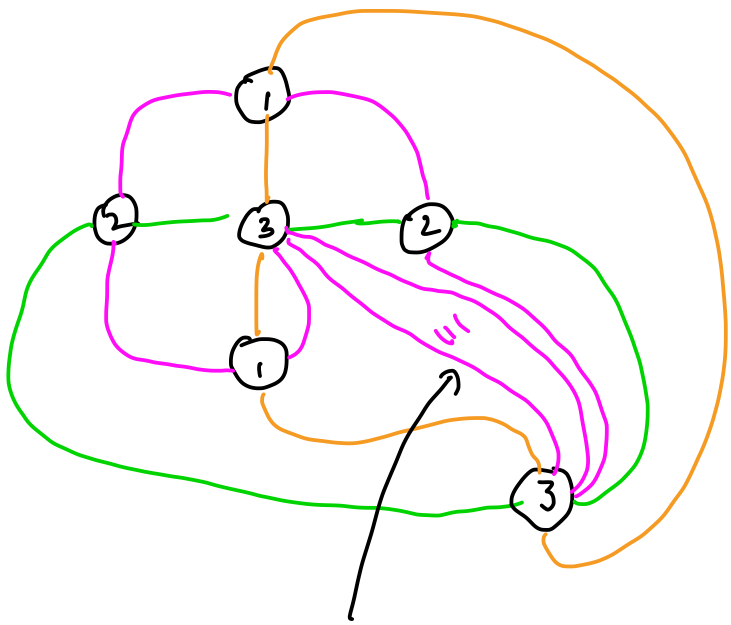


$(\theta, \phi) \mapsto (\theta, \phi + n\theta)$

so Y is determined by the genus g of Σ and an integer $n \in \mathbb{Z}$

denote Y by $Y_{g,n}$

exercise: Show $Y_{g,n}$ has a Heegaard diagram



exercise: Show $\Sigma_{0,n}$ is a lens space. Identify the lens space.

twisted bundles:

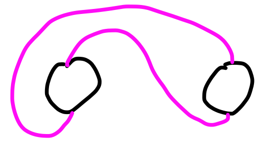
if N is a non oriented surface then

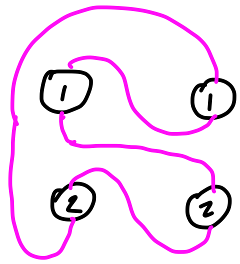
$$N \times I \quad \text{and} \quad N \times S^1$$

are non-oriented 3-manifolds

but we can take twisted I and S^1-bundles to get oriented 3-manifolds

exercise: 1) show twisted I-bundles have diagrams



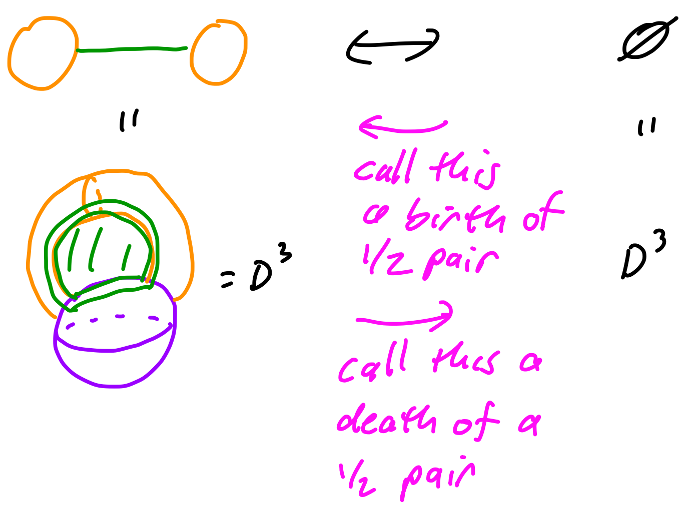


2) draw pictures for twisted S^1-bundles

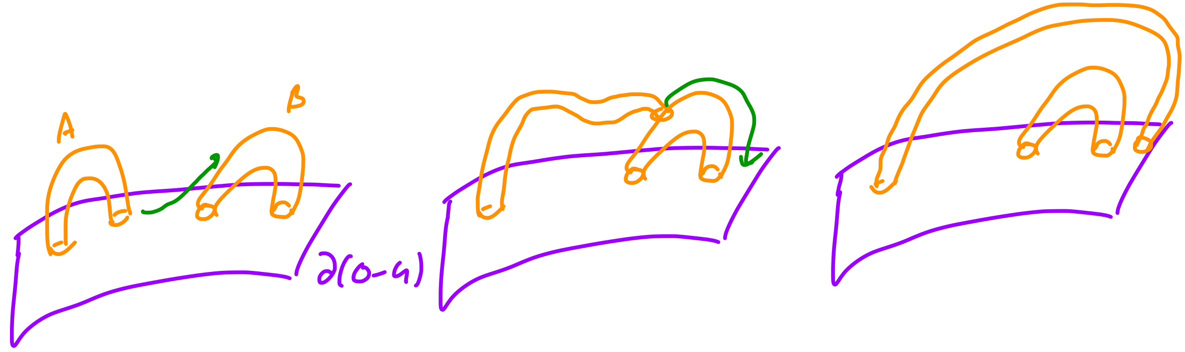
3) Show one of the twisted S^1-bundles over $\mathbb{R}P^2$ is a connected sum

B. Moves between Heegaard diagrams

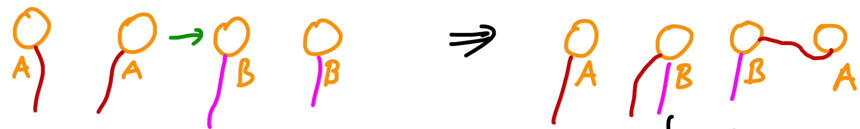
as discussed in Section I we have cancelling 1,2-pairs



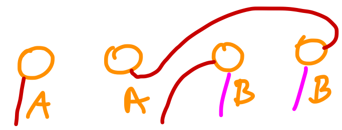
we also have handle slides for 1-handles



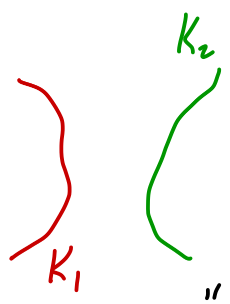
in the Kirby diagram we see



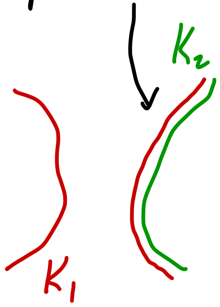
↓ can isotop back to



for 2-handles:

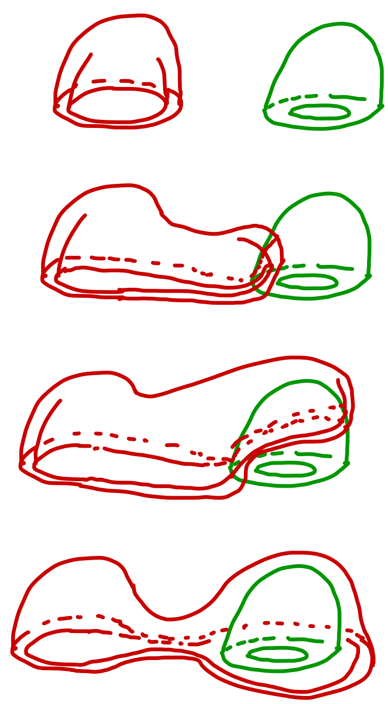


take a "push off" of K_2

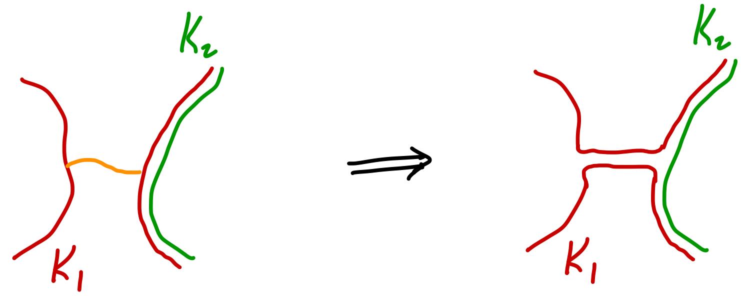


choose arc K_1 to push off

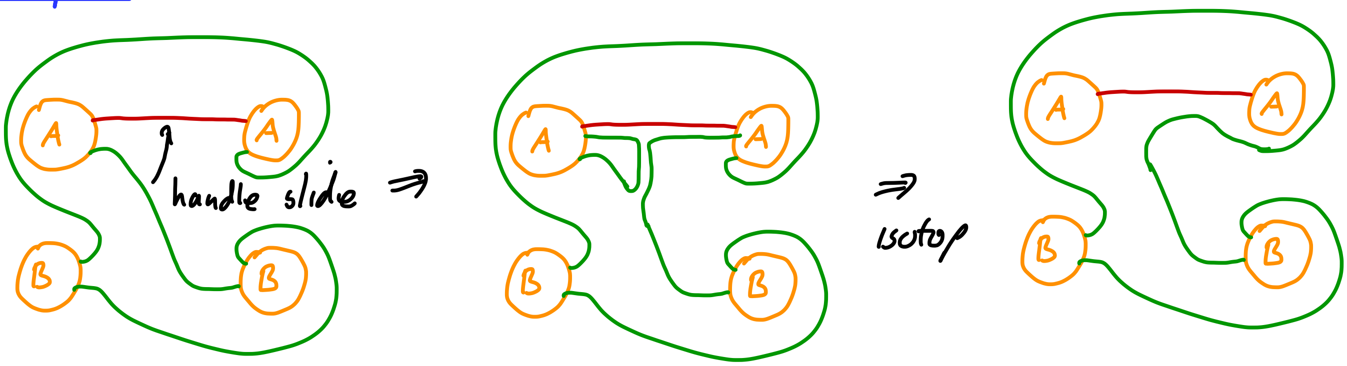
to push K_1 over K_2

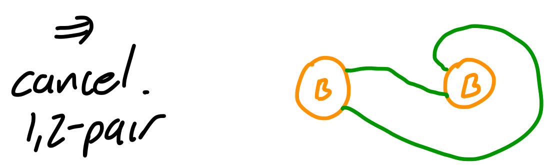


take an arc
to push K_1 along



example:





this is $L(2,1) \cong \mathbb{R}P^3$

exercise: above you were asked to show a twisted S^1-bundle over $\mathbb{R}P^2$ was a connect sum.
draw a Heegaard diagram for this bundle and use the above manipulations to show it is a connected sum

Theorem 1:

any two Heegaard diagrams of a fixed 3-manifold are related by a sequence of birth/deaths of 1/2 pairs and handle slides

we do not prove this now, but will see it follows from general facts about handlebodies we want to consider another version of this theorem

given a Heegaard surface $\Sigma \subset M^3$

let α be an arc embedded in Σ

let β be the result of isotoping interior of α off Σ

so $\alpha \cup \beta = \partial D^2$ s.t. $D^2 \cap \Sigma = \alpha$

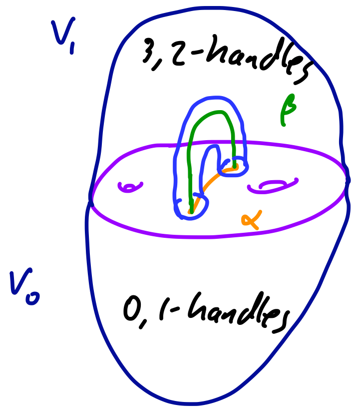
note $\beta \subset$ one component, say V_1, of $M \setminus \Sigma = V_0 \cup V_1$

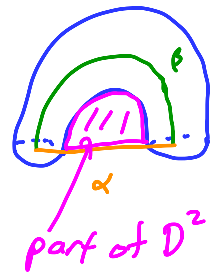
let $N = \text{nbhd } \beta$ in $V_1 = D^2 \times [0,1]$

let $V_1' = \overline{V_1 - N}$, $V_0' = V_0 \cup N$

<u>claim</u>: $\Sigma' = \partial V_0'$ a new Heegaard splitting of M

indeed





so we add a 1-handle to V_0

so V_0' still handle body

then attach nbhd of D^2 to V_0'

(i.e. 2-handle) and now

add the other 2-handles & 3-handle

i.e. $V_1' =$ 3-handle $\cup$ 2-handles

so handlebody

going from Σ to Σ'

is called a stabilization

note on handle decomposition level we just added a cancelling pair of handles

Theorem 2 (Reidemeister-Singer):

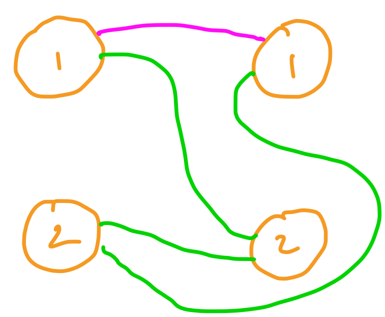
> any 2 Heegaard surfaces for M are isotopic after possibly stabilizing each surface

we prove Th$^{\underline{m}}$ 1 and 2 later (need Cerf theory)

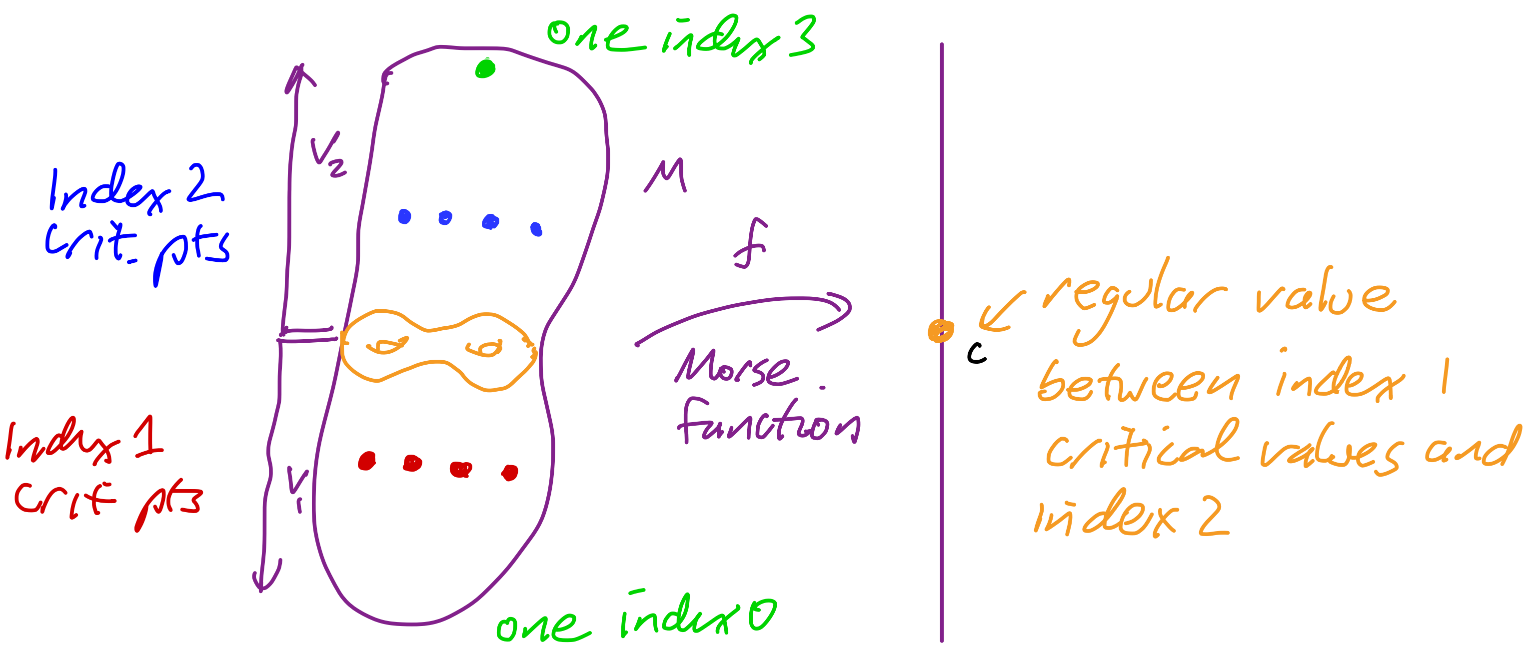
We now revisit Heegaard diagrams

above a Heegaard digram built M by attaching 1-handles to B^3 and then 2-handles

eg



but we can build M from the "middle out"



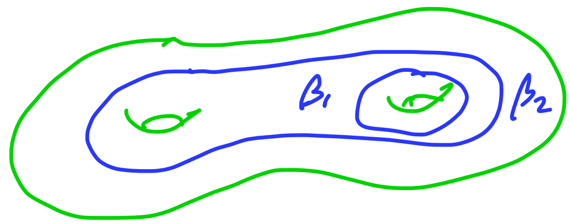
let Σ = level set of f between index 1 and 2 critical points $= f^{-1}(c)$

we can start with $\Sigma \times [0,1]$ (a neighborhood of Σ)

note $V_2 = f^{-1}([c,\infty))$ is built by attaching

2-handles to attaching curves and then a 3-handle

i.e. V_2 =

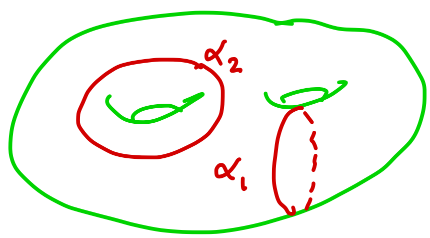


<u>note</u>: to attach a 3-handle need g attaching curves

<u>exercise</u>: show if $\Sigma \setminus \bigcup \beta_i$ is connected then can attach 3-handle

now think of V_1 upside down so the 1-handles become 2-handles attached along the belt spheres of 1-handles

V_1 =



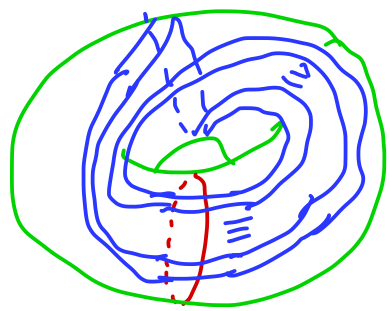
So given
- genus g surface Σ
- g disjoint curves $\alpha_1 \dots \alpha_g$ on Σ s.t. $\Sigma \setminus \bigcup \alpha_i$ is connected
- g disjoint curves $\beta_1 - - \beta_g$ on Σ s.t. $\Sigma \setminus \bigcup \beta_i$ is connected

we can construct a closed manifold and any closed 3-manifold can be built this way!

we call $(\Sigma_g, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1 \dots \beta_g\})$ a Heegaard diagram

examples:

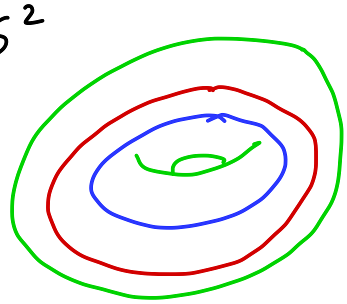
1) Lens space



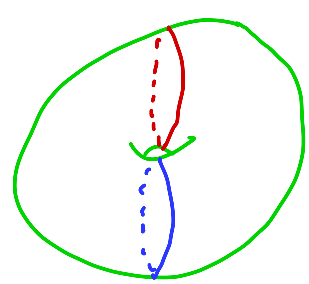
exercise: if α is a (r,s) curve on T^2
β is a (n,m) " "

then (T^2, α, β) is a lens space
which one?

2) $S^1 \times S^2$



and



exercise: draw such Heegaard diagrams for all the manifolds we discussed above